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The quantum field theory of fermion mixing

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Abstract. Blasone and Vitiello showed that for two or three flavours the mass and flavour vacua for fermion mixing define inequivalent representations of the anticommutation relations. This paper presents a short proof that this holds for any number of flavours and momentum-dependent mixing, and also extends the Blasone–Vitiello formulae for the oscillations.

It has been known for some time that otherwise identical fermions with different masses give rise to oscillations [1], and recent experiments have provided evidence in support of this in the case of solar neutrinos [4–6]. Blasone and Vitiello recently pointed out that, far from being approximately equal (as often assumed in theoretical treatments), the flavour and Dirac vacua for the Pontecorvo theory of fermion mixing generate inequivalent representations of the fermionic anticommutation relations [2, 3]. This paper applies standard mathematical criteria [7, 9–11] to give a short proof of a generalization of their result to arbitrary numbers of flavours, with a mixing matrix which can be momentum dependent. (Although there is currently no evidence for more than three generations of leptons, the proof still has the advantage that the cases of two and three flavours are treated in a unified way.)

The fermionic anticommutation relations (CAR) take the form

$$[a(\xi)^*, a(\eta)]_+ = \langle \xi, \eta \rangle$$

where ξ and η lie in the space $\mathcal{H}$ of wavefunctions on $\mathbb{R}^3$ with values in the product $\mathcal{V}$ of the Dirac spinors and an N-dimensional space V describing the various flavour states. The Pontecorvo mixing operator is given by a unitary operator T on V. The Dirac vacuum $\Omega \in \mathcal{H}$ satisfies the condition

$$a(P_+\xi)\Omega = 0 = a(P_-\xi)^*\Omega$$

where P_+ and P_- are the projections onto the positive and negative energy states, respectively. The vectors in $\mathcal{H}$ can be regarded as initial data for a Dirac equation of the form

$$\mathrm{i}\hbar \partial_t \psi = H_D \psi$$

where the Dirac–Hamiltonian has the form

$$H_D = c(\boldsymbol{\alpha} \cdot \boldsymbol{P} \otimes 1 + \beta \otimes Mc)$$

with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β satisfying the Clifford algebra relations

$$\alpha_j \beta + \beta \alpha_j = 0 \qquad \beta^2 = 1 \qquad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} \qquad j, k = 1, 2, 3$$

and M is a positive operator on V, with eigenvalues the masses $m_1, m_2, \ldots, m_N$ of the variously flavoured particles. (We shall generally omit the tensor products and simply write βM for $\beta \otimes M$, etc.)

We readily check that

$$H_D^2 = \left(M^2c^4 + c^2|\boldsymbol{P}|^2\right).$$

Defining

$$P_\pm = \tfrac{1}{2}\left(1 \pm H_D(M^2c^4 + c^2|\boldsymbol{P}|^2)^{-1/2}\right)$$

where $(M^2c^4 + c^2|\boldsymbol{P}|^2)^{-1/2}$ is the inverse of the positive operator square root of H_D^2, we now readily check that $P_\pm^2 = P_\pm$ and that $H_D P_\pm = \pm(M^2c^4 + c^2|\boldsymbol{P}|^2)^{1/2}P_\pm$, so that $P_\pm$ are the positive and negative energy projections.

By Fourier transforming the wavefunctions into the momentum space representation, $\mathcal{H}$ decomposes into a direct integral of spaces $\mathcal{H}_{\boldsymbol{p}}$, one for each momentum $\boldsymbol{p} \in \mathbb{R}^3$, on which the positive and negative energy projections are

$$P_\pm = \tfrac{1}{2}(1 \pm (c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta Mc^2)(M^2c^4 + c^2|\boldsymbol{p}|^2)^{-1/2}).$$

The question of equivalence of the Dirac and flavour vacua can be reduced to the question of whether the Pontecorvo operator T can be implemented in the representation of the CAR generated by Ω, and this occurs if and only if the operator P_+TP_- is a Hilbert–Schmidt operator. In general, as we shall now show, it is not. This is easily seen by noting that P_+TP_- acts on $\mathcal{H}_{\boldsymbol{p}}$ just as a multiplication by a matrix $F(\boldsymbol{p})$. (This is still true when T depends on the momentum, and we shall allow for this possibility in what follows.) Similarly, $P_-T^*P_+TP_-$ is multiplication by $F(\boldsymbol{p})^*F(\boldsymbol{p})$ and so can be represented as an integral operator with kernel $k(\boldsymbol{p},\boldsymbol{q}) = F(\boldsymbol{p})^*F(\boldsymbol{p})\delta(\boldsymbol{p}-\boldsymbol{q})$. The trace $\mathrm{tr}_{\mathcal{H}}(P_-T^*P_+TP_-)$ is the integral of $\mathrm{tr}_{\mathcal{V}}(k(\boldsymbol{p},\boldsymbol{p}))$ and so clearly diverges unless $\mathrm{tr}_{\mathcal{V}}(F(\boldsymbol{p})^*F(\boldsymbol{p}))$ vanishes identically.

With respect to a basis of eigenvectors for M in V, M is represented by a diagonal matrix and T has matrix elements T_{jk}, say. The contribution to $\mathrm{tr}_{\mathcal{H}}(P_-T^*P_+TP_-)$ coming from $\mathcal{H}_{\boldsymbol{p}}$ is then found to be

$$\mathrm{tr}_{\mathcal{V}}(F(\boldsymbol{p})^*F(\boldsymbol{p})) = \tfrac{1}{4}\sum_{j,k=1}^{N}\mathrm{tr}_{\mathcal{V}}[(1 + (c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m_jc^2)E_j^{-1})(1 - (c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m_kc^2)E_k^{-1})]|T_{jk}|^2$$

where $E_j = (m_j^2c^4 + c^2|\boldsymbol{p}|^2)^{1/2}$. Using the fact that the matrices α_j have a trace of zero, and that the sum over the spinor degrees of freedom cancels the factor of $\frac{1}{4}$, this can be written as

$$\sum_{j,k=1}^{N}[1 - (c^2|\boldsymbol{p}|^2 + m_jm_kc^4)(E_jE_k)^{-1}]|T_{jk}|^2.$$

For brevity we write

$$S_{jk} = \frac{c^2|\boldsymbol{p}|^2 + m_jm_kc^4}{E_jE_k}$$

so that

$$\mathrm{tr}_{\mathcal{V}}(F(\boldsymbol{p})^*F(\boldsymbol{p})) = \sum_{j,k=1}^{N}[1 - S_{jk}]|T_{jk}|^2$$

and, since $S_{jj} = 1$, the sum could be taken over $j \neq k$.

We now note that

$$(E_j E_k)^2 - (c^2|\boldsymbol{p}|^2 + m_j m_k c^4)^2 = (c^2|\boldsymbol{p}|^2 + m_j^2 c^4)(c^2|\boldsymbol{p}|^2 + m_k^2 c^4) - (c^2|\boldsymbol{p}|^2 + m_j m_k c^4)^2$$
$$= (m_j - m_k)^2 c^6 |\boldsymbol{p}|^2$$

so that

$$1 - S_{jk} = \frac{1 - S_{jk}^2}{1 + S_{jk}} = \frac{(m_j - m_k)^2 c^6 |\boldsymbol{p}|^2}{(1 + S_{jk})(E_j E_k)^2}$$

from which it follows that the terms in the sum are all non-negative and so $\mathrm{tr}_{\mathcal{V}}(F(\boldsymbol{p})^* F(\boldsymbol{p}))$ vanishes if and only if each term vanishes, which is equivalent to

$$(m_j - m_k)^2 |T_{jk}|^2 = 0$$

for all j and k. We conclude that T is implementable if and only if $T_{jk} = 0$ whenever $m_j \neq m_k$, i.e. there is no mixing of different masses.

By differentiating we can readily show that the maximum value of $1 - S_{jk}^2$, which occurs when $p^2 = m_j m_k c^2$, is

$$\left(\frac{m_j - m_k}{m_j + m_k}\right)^2.$$

This gives $(4m_j m_k)/(m_j + m_k)^2 \leqslant S_{jk}^2 \leqslant 1$, and since S_{jk} is positive,

$$\frac{2\sqrt{m_j m_k}}{m_j + m_k} \leqslant S_{jk} \leqslant 1.$$

These inequalities can also be written as

$$0 \leqslant 1 - S_{jk} \leqslant \frac{(\sqrt{m_j} - \sqrt{m_k})^2}{m_j + m_k}.$$

We shall now consider oscillations for fermions of a fixed energy, and we start by recalling the formulae for correlation functions. Let $\widetilde{A}$ be the second quantization of a single-particle operator A (which satisfies $[\widetilde{A}, a(\phi)^*] = a(A\phi)^*$ for any single-particle state ϕ, and also $\widetilde{A}\Omega = 0$). Then, using the anticommutation relations we have

$$\begin{aligned}\langle a(\phi)^*\Omega, \widetilde{A}a(\phi)^*\Omega\rangle &= \langle a(P_+\phi)^*\Omega, \widetilde{A}a(P_+\phi)^*\Omega\rangle \\ &= \langle \Omega, a(P_+\phi)(a(AP_+\phi)^* + a(P_+\phi)^*\widetilde{A})\Omega\rangle \\ &= \langle P_+\phi, AP_+\phi\rangle = \langle \phi, P_+AP_+\phi\rangle.\end{aligned}$$

(Similarly, $\|a(\phi)^*\Omega\|^2 = \langle \phi, P_+\phi\rangle$.) Generalizing this to combinations of vectors ϕ, we consider expectations of the form $\mathrm{tr}_{\mathcal{H}}(BP_+AP_+)/\,\mathrm{tr}_{\mathcal{H}}(BP_+)$ for positive operators B. Now the number operator for particles of flavour κ is the second quantization of the projection operator P^κ onto the subspace of flavour κ in V (or, strictly speaking, of $1 \otimes P_\kappa$). Taking $B = P^\lambda$ we see that the expected number of particles of flavour κ in the state obtained by applying λ-flavour creation operators to the Dirac vacuum is $\mathrm{tr}_{\mathcal{H}}(P^\lambda P_+ P^\kappa P_+)/\,\mathrm{tr}_{\mathcal{H}}(P^\lambda P_+)$. This can be evaluated in a similar way to our earlier calculations (apart from a sign change in one of the projections).

The numerator is thus found to be

$$\sum_{j,k=1}^{N} (1 + S_{jk}) P_{jk}^\lambda P_{kj}^\kappa.$$

Moreover, by the definition of T, we see that, if the flavour states are non-degenerate, the projection P^λ is the conjugate of projection onto the λth basis vector and has components

$$P^\lambda_{jk} = T^*_{j\lambda} T_{\lambda k} = \overline{T}_{\lambda j} T_{\lambda k}$$

giving

$$\sum_{j,k=1}^{N} (1 + S_{jk}) \overline{T}_{\lambda j} T_{\lambda k} \overline{T}_{\kappa k} T_{\kappa j}.$$

By the unitarity of T we have $\sum_j \overline{T}_{\kappa j} T_{\lambda j} = \delta_{\kappa\lambda}$, so that the numerator can be rewritten as

$$2\delta_{\kappa\lambda} - \sum_{j,k=1}^{N} (1 - S_{jk}) \overline{T}_{\lambda j} T_{\lambda k} \overline{T}_{\kappa k} T_{\kappa j}.$$

The denominator is just

$$\mathrm{tr}_{\mathcal{H}}(P^\lambda P_+) = 2 \sum_{j=1}^{N} P^\lambda_{jj} = 2 \sum_{j=1}^{N} |T_{\lambda j}|^2 = 2$$

so that when $\kappa = \lambda$ the expectation value is

$$1 - \tfrac{1}{2} \sum_{j \neq k} (1 - S_{jk}) |T_{\lambda j}|^2 |T_{\lambda k}|^2.$$

The second term represents the quantum field-theoretic correction. Our previous calculation shows that the correction to the j, k term is largest when $p^2 = m_j m_k c^2$, and provides bounds. We note that in the two-dimensional case considered in detail by Blasone and Vitiello the numerator of the correction collapses to a single term.

To incorporate the time development we note that in the Heisenberg picture P^κ evolves in time t to

$$P^\kappa(t) = \mathrm{e}^{\mathrm{i}t H_D/\hbar} P^\kappa \, \mathrm{e}^{-\mathrm{i}t H_D/\hbar}$$

with matrix components

$$(P^\kappa(t))_{jk} = \mathrm{e}^{\mathrm{i}t(E_j - E_k)/\hbar} (P^\kappa)_{jk}.$$

When this is substituted in place of $(P^\kappa)_{jk}$ the denominator is unchanged, but we obtain for the numerator

$$\sum_{j,k=1}^{N} (1 + S_{jk}) \, \mathrm{e}^{\mathrm{i}t(E_k - E_j)/\hbar} |T_{\kappa j}|^2 |T_{\kappa k}|^2.$$

Exploiting the symmetry in j and k this becomes

$$\sum_{j,k=1}^{N} (1 + S_{jk}) \cos(t(E_k - E_j)/\hbar) |T_{\kappa j}|^2 |T_{\kappa k}|^2.$$

This expression differs from that at time $t = 0$ by

$$2 \sum_{j,k=1}^{N} \sin^2(t(E_k - E_j)/2\hbar)](1 + S_{jk}) |T_{\kappa j}|^2 |T_{\kappa k}|^2$$

exhibiting the oscillations.

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